



MARKSCHEME

November 2012

MATHEMATICS SETS, RELATIONS AND GROUPS

Higher Level

Paper 3

12 pages

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a correct **Method**; working must be seen.
- (M)** Marks awarded for **Method**; may be implied by **correct** subsequent working.
- A** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding **M** marks.
- (A)** Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.
- R** Marks awarded for clear **Reasoning**.
- N** Marks awarded for **correct** answers if **no** working shown.
- AG** Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to scoris instructions and the document “**Mathematics HL: Guidance for e-marking Nov 2012**”. It is **essential** that you read this document before you start marking. In particular, please note the following.

Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.

- If a part is **completely correct**, (and gains all the “must be seen” marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp **A0** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.

All the marks will be added and recorded by scoris.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award **M0** followed by **A1**, as **A** mark(s) depend on the preceding **M** mark(s), if any.
- Where **M** and **A** marks are noted on the same line, *e.g.* **M1A1**, this usually means **M1** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and **A1** for using the **correct** values.
- Where the markscheme specifies **(M2)**, **N3**, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 **N marks**

Award **N** marks for **correct** answers where there is **no** working.

- Do **not** award a mixture of **N** and other marks.
- There may be fewer **N** marks available than the total of **M**, **A** and **R** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 **Implied marks**

Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 **Follow through marks**

Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent part(s)**. To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.

- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks.
- If the error leads to an inappropriate value (e.g. $\sin \theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent A** marks can be awarded, but **M** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 **Mis-read**

If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a mis-read. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (e.g. $\sin \theta = 1.5$), do not award the mark(s) for the final answer(s).

7 **Discretionary marks (d)**

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation **DM** should be used and a brief **note** written next to the mark explaining this decision.

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for part-questions are indicated by **EITHER . . . OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x - 3)$, the markscheme gives:

$$f'(x) = (2\cos(5x - 3))5 \quad (= 10\cos(5x - 3)) \quad \mathbf{AI}$$

Award **AI** for $(2\cos(5x - 3))5$, even if $10\cos(5x - 3)$ is not seen.

10 Accuracy of Answers

Candidates should **NO LONGER** be penalized for an accuracy error (**AP**).

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (e.g. TI-89) are not allowed.

Calculator notation

The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) (i) not reflexive *e.g.* $1+1=2$ **R1**
- (ii) symmetric since $x+y=y+x$ **R1**
- (iii) *e.g.* $1+11>7$, $11+2>7$ but $1+2=3$, so not transitive **M1A1**
- [4 marks]**
- (b) (i) reflexive since $x-x=0$ **R1**
- (ii) symmetric since $|x-y|=|y-x|$ **R1**
- (iii) *e.g.* $1R2$, $2R3$ but $1-3=-2$, so not transitive **M1A1**
- [4 marks]**
- (c) (i) reflexive since $x^2 > 0$ **R1**
- (ii) symmetric since $xy = yx$ **R1**
- (iii) $xy > 0$ and $yz > 0 \Rightarrow xy^2z > 0 \Rightarrow xz > 0$ since $y^2 > 0$, so transitive **M1A1**
- [4 marks]**
- (d) (i) reflexive since $\frac{x}{x} = 1$ **R1**
- (ii) not symmetric *e.g.* $\frac{2}{1} = 2$ but $\frac{1}{2} = 0.5$ **R1**
- (iii) $\frac{x}{y} \in \mathbb{Z}$ and $\frac{y}{z} \in \mathbb{Z} \Rightarrow \frac{xy}{yz} = \frac{x}{z} \in \mathbb{Z}$, so transitive **M1A1**
- [4 marks]**

Note: For each **R1** mark the correct decision and a valid reason must be given.

- (e) only (c) is an equivalence relation **(A1)**
 the equivalence classes are **A1A1**
 $\{1, 2, 3, \dots\}$ and $\{-1, -2, -3, \dots\}$ **[3 marks]**

Total [19 marks]

2. (a) $f\left(\begin{pmatrix} 5 \\ 6 \end{pmatrix}\right) = \begin{pmatrix} 17 \\ 39 \end{pmatrix}$ *A1*
[1 mark]

(b) let $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\det M = -2$ so M^{-1} exists *M1*
 $f\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = f\left(\begin{pmatrix} c \\ d \end{pmatrix}\right) \Rightarrow M\begin{pmatrix} a \\ b \end{pmatrix} = M\begin{pmatrix} c \\ d \end{pmatrix} \Rightarrow M^{-1}M\begin{pmatrix} a \\ b \end{pmatrix} = M^{-1}M\begin{pmatrix} c \\ d \end{pmatrix}$
 $\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$ *R1*
 thus f is an injection *AG*
[2 marks]

(c) for any $\begin{pmatrix} x \\ y \end{pmatrix}$ we have to find an $\begin{pmatrix} a \\ b \end{pmatrix}$ so that $f\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = \begin{pmatrix} x \\ y \end{pmatrix}$
 M^{-1} exists $\left(= \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{pmatrix} \right)$ *M1*
 $M\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = M^{-1}\begin{pmatrix} x \\ y \end{pmatrix} \left(= \begin{pmatrix} -2x + y \\ \frac{3}{2}x - \frac{1}{2}y \end{pmatrix} \right)$ *R1*
 thus f is a surjection *AG*
[2 marks]

(d) $f^{-1}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -2x + y \\ \frac{3}{2}x - \frac{1}{2}y \end{pmatrix}$ *(M1)A1*

Note: Allow answer in matrix form.

[2 marks]

(e) $\det\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = 0$ *M1*
 so this matrix does not have an inverse, g does not have an inverse function
 and so g is not a bijection *A1*

Note: Finding counterexample showing that g is not injective (or alternatively, not surjective) is acceptable.

[2 marks]

Total [9 marks]

3. (a) $\emptyset, \{a\}, \{b\}, \{a, b\}$

A1
[1 mark]

(b)

Δ	\emptyset	$\{a\}$	$\{b\}$	$\{a, b\}$
\emptyset	\emptyset	$\{a\}$	$\{b\}$	$\{a, b\}$
$\{a\}$	$\{a\}$	\emptyset	$\{a, b\}$	$\{b\}$
$\{b\}$	$\{b\}$	$\{a, b\}$	\emptyset	$\{a\}$
$\{a, b\}$	$\{a, b\}$	$\{b\}$	$\{a\}$	\emptyset

A3

Note: Award *A2* for one error, *A1* for two errors, *A0* for more than two errors.

[3 marks]

(c) closure is seen from the table above
 \emptyset is the identity
 each element is self-inverse

A1
A1
A1

Note: Showing each element has an inverse is sufficient.

associativity is assumed so we have a group

AG

[3 marks]

(d) not isomorphic as in the above group all elements are self-inverse
 whereas in $(\mathbb{Z}_4, +_4)$ there is an element of order 4 (e.g. 1)

R2

[2 marks]

(e) (i) \emptyset is the identity

A1

(ii) $X^{-1} = X$

A1

(iii) if X and Y are subsets of S then $X\Delta Y$ (the set of elements that belong to X or Y but not both) is also a subset of S , hence closure is proved

R1

$\{P(S), \Delta\}$ is a group because it is closed, has an identity, all elements have inverses (and Δ is associative)

RIAG

[4 marks]

(f) not a group because although the identity is \emptyset , if $X \neq \emptyset$ it is impossible to find a set Y such that $X \cup Y = \emptyset$, so there are elements without an inverse

RIAG

[1 mark]

(g) not a group because although the identity is S , if $X \neq S$ it is impossible to find a set Y such that $X \cap Y = S$, so there are elements without an inverse

RIAG

[1 mark]

Total [15 marks]

4. (a) $\frac{c}{2} * \frac{3c}{4} = \frac{\frac{c}{2} + \frac{3c}{4}}{1 + \frac{1}{2} \cdot \frac{3}{4}}$ *M1*

$= \frac{\frac{5c}{4}}{\frac{11}{8}} = \frac{10c}{11}$ *A1*

[2 marks]

(b) identity is 0 *A1*

[1 mark]

(c) inverse is $-x$ *A1*

[1 mark]

(d) $x * y = \frac{x + y}{1 + \frac{xy}{c^2}}, y * x = \frac{y + x}{1 + \frac{yx}{c^2}}$ *M1*

(since ordinary addition and multiplication are commutative)

$x * y = y * x$ so $*$ is commutative *R1*

Note: Accept arguments using symmetry.

[2 marks]

continued ...

Question 4 continued

$$(e) \quad (x * y) * z = \frac{x + y}{1 + \frac{xy}{c^2}} * z = \frac{\left(\frac{x + y}{1 + \frac{xy}{c^2}}\right) + z}{1 + \left(\frac{x + y}{1 + \frac{xy}{c^2}}\right) \frac{z}{c^2}} \quad \text{M1}$$

$$= \frac{\left(x + y + z + \frac{xyz}{c^2}\right)}{\left(1 + \frac{xy}{c^2}\right)} = \frac{\left(x + y + z + \frac{xyz}{c^2}\right)}{\left(1 + \left(\frac{xy + xz + yz}{c^2}\right)\right)} \quad \text{A1}$$

$$\frac{\left(x + \frac{xyz}{c^2} + y + z\right)}{\left(1 + \frac{xy}{c^2}\right)}$$

$$x * (y * z) = x * \left(\frac{y + z}{1 + \frac{yz}{c^2}}\right) = \frac{x + \left(\frac{y + z}{1 + \frac{yz}{c^2}}\right)}{1 + \frac{x}{c^2} \left(\frac{y + z}{1 + \frac{yz}{c^2}}\right)}$$

$$= \frac{\left(x + \frac{xyz}{c^2} + y + z\right)}{\left(1 + \frac{yz}{c^2}\right)} = \frac{\left(x + y + z + \frac{xyz}{c^2}\right)}{\left(1 + \left(\frac{xy + xz + yz}{c^2}\right)\right)} \quad \text{A1}$$

$$\frac{\left(x + \frac{xyz}{c^2} + y + z\right)}{\left(1 + \frac{yz}{c^2}\right)}$$

since both expressions are the same * is associative R1

Note: After the initial *MIAI*, correct arguments using symmetry also gain full marks.

[4 marks]

(f) (i) $c > x$ and $c > y \Rightarrow c - x > 0$ and $c - y > 0 \Rightarrow (c - x)(c - y) > 0$ RIAG

(ii) $c^2 - cx - cy + xy > 0 \Rightarrow c^2 + xy > cx + cy \Rightarrow c + \frac{xy}{c} > x + y$ (as $c > 0$)

so $x + y < c + \frac{xy}{c}$ MIAG

[2 marks]

continued ...

Question 4 continued

- (g) if $x, y \in G$ then $-c - \frac{xy}{c} < x + y < c + \frac{xy}{c}$
 thus $-c \left(1 + \frac{xy}{c^2}\right) < x + y < c \left(1 + \frac{xy}{c^2}\right)$ and $-c < \frac{x + y}{1 + \frac{xy}{c^2}} < c$ **M1**
- (as $1 + \frac{xy}{c^2} > 0$) so $-c < x * y < c$ **A1**
 proving that G is closed under $*$ **AG**
- [2 marks]**
- (h) as $\{G, *\}$ is closed, is associative, has an identity and all elements
 have an inverse **R1**
 it is a group **AG**
 as $*$ is commutative **R1**
 it is an Abelian group **AG**
- [2 marks]**
- (i) as $c \rightarrow \infty, \{G, *\} \rightarrow \{\mathbb{R}, +\}$ **A1**
- [1 mark]**
- Total [17 marks]**